Diamond Triangulations Contain Spanners of Bounded Degree

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Abstract

Given a triangulation G, whose vertex set V is a set of n points in the plane, and given a real number γ with $0 < \gamma < \pi$, we design an $O(n)$-time algorithm that constructs a connected spanning subgraph G' of G whose maximum degree is at most $14 + \lceil 2\pi/\gamma \rceil$. If G is the Delaunay triangulation of V, and $\gamma = 2\pi/3$, we show that G' is a t-spanner of V (for some constant t) with maximum degree at most 17, thereby improving the previously best known degree bound of 23. If G is a triangulation satisfying the diamond property, then for a specific range of values of γ dependent on the angle of the diamonds, we show that G' is a t-spanner of V (for some constant t) whose maximum degree is bounded by a constant dependent on γ. If G is the graph consisting of all Delaunay edges of length at most 1, and $\gamma = \pi/3$, we show that a modified version of the algorithm produces a plane subgraph G' of the unit-disk graph which is a t-spanner (for some constant t) of the unit-disk graph of V, whose maximum degree is at most 20, thereby improving the previously best known degree bound of 25.

1 Introduction

Let V be a set of n points in the plane and let $t \geq 1$ be a real number. An undirected graph G with vertex set V is called a *t-spanner* of V, if for any two vertices u and v of V, G contains a path between u and v, whose length is at most $t|uv|$, where $|uv|$ denotes the Euclidean distance between u and v. The *stretch factor* (or *dilation*) of G is defined to be the smallest value of t for which G is a t-spanner of V.

The problem of constructing a t-spanner with $O(n)$ edges for any given point set has been studied intensively; see the book by Narasimhan and Smid [10].

In this paper, we focus on spanners that are *plane*, i.e., the interiors of any two (straight-line) edges of the spanner are disjoint. Chew [4] and Dobkin *et al.* [6] were the first to show the existence of plane spanners. Dobkin *et al.* proved that the *Delaunay triangulation* of V is a t-spanner of V, for $t = ((1+\sqrt{5})/2)\pi$. Keil and Gutwin [7] improved the analysis, and

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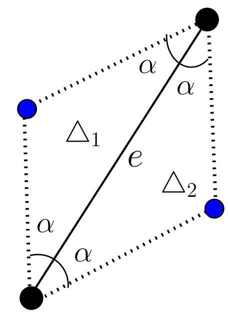

Figure 1: *An illustration of the diamond property. At least one of the triangles* Δ_1 *and* Δ_2 *does not contain any point of* V*.*

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showed that the Delaunay triangulation is a t-spanner for $t = \frac{4\pi\sqrt{3}}{9}$. A more general result appears in Bose *et al.* [2]: For every two vertices u and v of V, the Delaunay triangulation contains a path between u and v of length at most $\frac{4\pi\sqrt{3}}{9} \cdot |uv|$, all of whose edges have length at most $|uv|$.

Das and Joseph [5] generalized these results to triangulations that satisfy the so-called *diamond property*: Let G be a triangulation of V, and let α be a real number with $0 < \alpha < \frac{\pi}{2}$. Let e be an edge of G, and consider the two isosceles triangles Δ_1 and Δ_2 with base e and base angle α; refer to Figure 1. We say that the edge e satisfies the α-*diamond property*, if at least one of Δ_1 and Δ_2 does not contain any point of V in its interior. We say that the triangulation G satisfies the α-*diamond property*, if every edge e of G satisfies this property.

Das and Joseph [5] showed that any triangulation satisfying the α-diamond property is a t-spanner, for some real number t that only depends on the value of α. (In fact, Das and Joseph considered plane graphs that, additionally, satisfy the so-called *good polygon property*.) The analysis was refined by Lee [8], who showed that $t \leq \frac{8(\pi-\alpha)^2}{\alpha^2 \sin^2(\alpha/4)}$. The Delaunay triangulation satisfies the α-diamond property for $\alpha = \pi/4$. Das and Joseph proved that both the greedy triangulation and the minimum weight triangulation satisfy the α-diamond property, for some constant α.

None of the results mentioned above lead to plane spanners in which the degree of every vertex is bounded by a constant. For example, the maximum vertex degree in a Delaunay triangulation can be $\Omega(n)$. Bose *et al.* [1] were the first to show the existence of a plane t-spanner (for some constant t), whose maximum vertex degree is bounded by a constant. To be more precise, they showed that the Delaunay triangulation of any set V of n points in the plane contains a subgraph, which is a t-spanner for V, where $t = \frac{4\pi(\pi+1)\sqrt{3}}{9}$, and whose maximum degree is at most 27. This result was improved by Li and Wang [9]: For any real number γ with $0 < \gamma \leq \pi/2$, the Delaunay triangulation contains a subgraph, which is a t-spanner, where $t = \max\{\frac{\pi}{2}, 1 + \pi \sin\frac{\gamma}{2}\} \cdot \frac{4\pi\sqrt{3}}{9}$, and whose maximum degree is at most $19 + \lceil 2\pi/\gamma \rceil$. For $\gamma = \pi/2$, the degree bound is 23. Given the Delaunay triangulation as input, both spanners of [1] and [9] can be computed in $O(n)$ time.

In this paper, we further improve the degree bound:

Theorem 1. *Let V be a set of n points in the plane, and let γ be a real number with $0 < \gamma \leq 2\pi/3$. Assume that we are given the Delaunay triangulation G of V. Then, in $O(n)$ time, we can compute a subgraph G' of G, such that G' is a t-spanner of V, where*

$$t = \begin{cases} \frac{4\pi\sqrt{3}}{9} \cdot \max\left\{\frac{\pi}{2}, 1 + \pi \sin\frac{\gamma}{2}\right\} & \text{if } \gamma < \pi/2, \\ \frac{4\pi\sqrt{3}}{9}\left(1 + 2\sqrt{3} + 3\pi/2 + \pi \sin\frac{\pi}{12}\right) & \text{if } \pi/2 \leq \gamma \leq 2\pi/3, \end{cases}$$

and the maximum degree of G' is at most $14 + \lceil 2\pi/\gamma \rceil$. Thus, for $\gamma = 2\pi/3$, the degree bound is 17.

We obtain this result by designing a linear-time algorithm that, when given an arbitrary triangulation G of the point set V, computes a spanning subgraph G' of G, that satisfies the degree bound in Theorem 1. We then show that when G is the Delaunay triangulation, the stretch factor of G' is at most the value of t in Theorem 1.

We also show that the algorithm in Theorem 1 can in fact be applied to any triangulation satisfying the diamond property:

Theorem 2. *Let V be a set of n points in the plane, let α be a real number with $0 < \alpha \leq \pi/2$, and assume that we are given a triangulation G of V that satisfies the α-diamond property. Then, in $O(n)$ time, we can compute a subgraph G' of G, such that G' is a t-spanner of V, where*

$$t = \left(1 + \frac{2(\pi - \alpha)}{\alpha \sin\frac{\alpha}{4}} \cdot \max\left\{1, 2\sin\frac{\alpha}{2}\right\}\right) \frac{8(\pi - \alpha)^2}{\alpha^2 \sin^2\frac{\alpha}{4}},$$

and the maximum degree of G' is at most $14 + \lceil 2\pi/\alpha \rceil$.

Thus, by combining Theorem 2 with the results of Das and Joseph [5], it follows that both the greedy triangulation and the minimum weight triangulation contain a t-spanner (for some constant t) whose maximum degree is bounded by a constant.

In the final part of the paper, We extend this result to the *unit-disk graph*, which is the graph where every two distinct points u and v in the vertex set are connected by an edge if and only if $|uv| \leq 1$. A t-spanner of the unit-disk graph is a subgraph of the unit-disk graph with the property that for every edge (u, v) of the unit-disk graph, there exists a path between u and v in the subgraph whose length is at most $t|uv|$. Let G be the graph consisting of all edges in the Delaunay triangulation of V, whose length is at most one. It follows from the result of Bose *et al.* [2] which was mentioned above, that G is a $\frac{4\pi\sqrt{3}}{9}$-spanner of the unit-disk graph of V. The construction for the Delaunay triangulation by Bose *et al.* [1] was modified by Li and Wang [9] to obtain a plane t-spanner (for some constant t) of the unit-disk graph whose maximum degree is at most 25. By modifying our algorithm, we obtain the following result:

Theorem 3. *Let V be a set of n points in the plane, and let γ be a real number with $0 < \gamma \leq \pi/3$. Assume that we are given the Delaunay triangulation G of V. Then, in $O(n)$ time, we can compute a plane graph G', such that G' is a t-spanner of the unit-disk graph of V, where*

$$t = \frac{4\pi\sqrt{3}}{9} \cdot \max\left\{\frac{\pi}{2}, 1 + \pi \sin\frac{\gamma}{2}\right\},$$

and the maximum degree of G' is at most $14 + \lceil 2\pi/\gamma \rceil$. Thus, for $\gamma = \pi/3$, the degree bound is 20.

We emphasize that the graph G' in Theorem 3 is not necessarily a subgraph of the Delaunay triangulation, but it is a plane subgraph of the unit-disk graph.

Bounded degree plane spanners of the unit-disk graph have applications in topology control of wireless ad hoc networks, see [9]. In such a network, a vertex u can only communicate with those vertices that are within the communication range of u. If we assume that this range is equal to one for each vertex, then the unit-disk graph models a wireless ad hoc network. Many algorithms for routing messages in such a network require that the underlying network topology is plane. Moreover, if the maximum degree is small, then the throughput of the network can be improved significantly.

The rest of the paper is organized as follows. In Section 2, we present the linear-time algorithm that computes a bounded degree subgraph of any given triangulation. In Section 3 and Section 4, we prove Theorems 1 and 2, respectively, whereas the proof of Theorem 3 is presented in Section 5. We conclude in Section 6 with some directions for future work.

2 Computing a bounded-degree subgraph of a triangulation

Let V be a set of n points in the plane, and let $G = (V, E)$ be a planar graph. We define a numbering of the elements of V in the following way: pick a vertex of minimum degree and assign it label 1. Then delete this vertex together with its incident edges, increment the label by 1 and recursively define a numbering of the remaining $n - 1$ vertices. The resulting numbering $(v_1, v_2, \ldots, v_n)$ of the vertex set V is called a *low-degree numbering.* The following lemma explains this terminology and presents an important property of a low-degree numbering.

Lemma 1. *Let v_i be a vertex in a low-degree numbering of V with index i, $1 \leq i \leq n$. The number of edges (v_i, v_j) of G, with $j > i$, is at most 5. Given the planar graph G, a low-degree numbering of V can be computed in $O(n)$ time.*

Proof. The first claim follows from the fact that every planar graph contains a vertex of degree at most 5.

A low-degree numbering can be computed by using an array $A[1 \ldots n]$, where $A[j]$ contains a list of all nodes having degree j. With each vertex w, we store a pointer to the occurrence of w in the list $A[j]$, where j is the degree of w.

In the i-th iteration of the algorithm, we pick the first vertex u in the subarray $A[1 \ldots 5]$ and label it v_i. Then we consider all edges (u, w) in the current graph. For any such edge (u, w), if w has degree j, then we delete w from the list $A[j]$, and insert it into list $A[j-1]$. Finally, we delete u from the subarray $A[1 \ldots 5]$. Since each node w will be updated at most $deg(w)$ times and each update takes constant time, the total running time is proportional to $\sum_w deg(w) \leq 2(3n - 6) = O(n)$. □

Algorithm BDEGSUBGRAPH(G, γ)

Input: A triangulation $G = (V, E)$ whose vertex set V is a set of n points in the plane, and a real number γ with $0 < \gamma < \pi$.

Output: A subgraph $G' = (V, E')$ of G whose maximum degree is at most $14 + \lceil \frac{2\pi}{\gamma} \rceil$.

1. compute a low-degree numbering $(v_1, v_2, \cdots, v_n)$ of $G = (V, E)$;
2. label each vertex of V as "unprocessed";
3. $E' = \emptyset$;
4. **for** $i = n$ **downto** 1
5.   **do if** v_i has "unprocessed" neighbors
6.   **then** compute the closest "unprocessed" neighbor x of v_i;
7.   divide the plane into cones $C_1, \ldots, C_{\lceil 2\pi/\gamma \rceil}$ with apex v_i and angle at most γ with segment $v_i x$ is on the boundary between C_1 and C_2;
8.   add the edge (v_i, x) to E'
9.   **else** go to line 18
10.   **for** each cone $C \notin \{C_1, C_2\}$
11.   **do** compute the closest "unprocessed" vertex w in $C \cap N_G(v_i)$;
12.   **if** w exists
13.   **then** add the edge (v_i, w) to E'
14.   let $w_0, w_1, \cdots, w_{d-1}$ be the clockwise order of the vertices in $N_G(v_i)$;
15.   **for** $k = 0$ **to** $d - 1$
16.   **if** w_k and $w_{(k+1) \bmod d}$ are both "unprocessed"
17.   **then** add the edge $(w_k, w_{(k+1) \bmod d})$ to E';
18.   label v_i as "processed";
19. **return** the graph $G' = (V, E')$

Figure 2: *The algorithm that computes a bounded-degree subgraph of a triangulation G.*

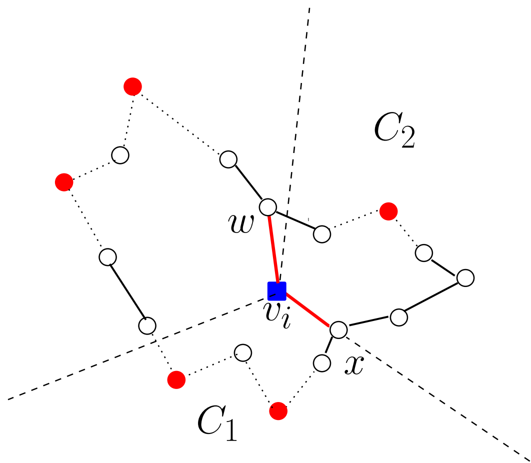


Figure 3: *An illustration of algorithm* BDEGSUBGRAPH*(G, γ), for $\gamma = 2\pi/3$, when processing vertex v_i. The figure shows v_i and all vertices in $N_G(v_i)$. Solid vertices represent the "processed" vertices, whereas hollow vertices represent the "unprocessed" vertices. When processing v_i, the algorithm adds the solid edges to the graph G'.*

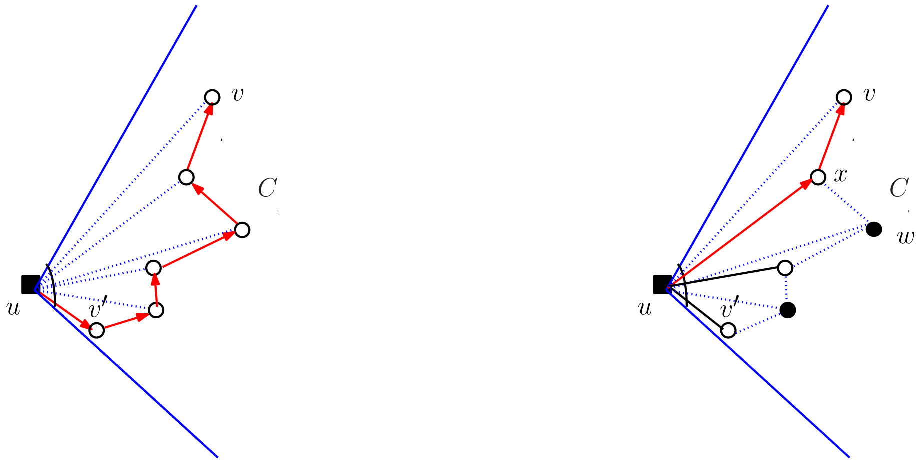
The algorithm that computes a bounded-degree subgraph of any given triangulation is given in Figure 2; for an illustration, refer to Figure 3. In this algorithm, $N_G(v)$ denotes the set of *neighbors* of the vertex v in G, i.e. $N_G(v) = \{w \in V : (v, w) \in E\}$.

We assume that the triangulation is stored in a doubly-connected edge list. Then, for any vertex v, we can obtain the vertices in $N_G(v)$, sorted in angular order around v, in time proportional to the degree of v. This observation, together with Lemma 1 and the fact that G contains $O(n)$ edges, implies that the running time of algorithm BDEGSUBGRAPH is $O(n)$.

Lemma 2. *Let G be a triangulation whose vertex set V is a set of n points in the plane, and let γ be a real number with $0 < \gamma < \pi$. Let G' be the graph that is returned by algorithm* BDEGSUBGRAPH*(G, γ). Then, G' is a connected spanning subgraph of G.*

Proof. G' is a spanning subgraph of G since its vertex set is V and the only edges added to E' are from E. Therefore, it suffices to show that for each edge (u, v) of $G \setminus G'$, the graph G' contains a path between u and v. Let (u, v) be such an edge of $G \setminus G'$. We may assume without loss of generality that u is processed before v. Let C be the cone with apex u and having angle at most γ that contains v and that is constructed when vertex u is processed. Let v' be the closest neighbor of u that is contained in C and that is unprocessed at the moment when u is processed. Thus, during the processing of u, the edge (u, v') is added to G'. We may assume without loss of generality that (u, v') is clockwise to the right of (u, v). Observe that $\angle vuv' \leq \gamma$.

We distinguish two cases.



(a) Case 1: All neighbors of u between v and v' are unprocessed.

(b) Case 2: At least one neighbor of u between v and v' has been processed.

Figure 4: *Illustrations of the proof of Lemma 2.*

Case 1: At the moment when u is processed, all neighbors of u between v and v' are unprocessed. Then by algorithm BDEGSUBGRAPH(G, γ), there is a path P_{uv} between u and v shown as a directed path through vertex v' in Figure 4(a).

Case 2: At the moment when u is processed, at least one neighbor of u between v and v' has been processed. Let w be the last processed neighbor of u in clockwise order of v in C, i.e., there is no processed neighbor of u between v and w in C. Let x be the first neighbor of u clockwise to the left of w (observe that x could be v). In Δ_{uwx}, w is processed before x and u. Since $\Delta_{uwx} \in G$, when processing w, by algorithm BDEGSUBGRAPH(G, γ), edge (u, x) must have been added into E'. Similar to Case 1, there is a path P_{uv} between u and v shown as a directed path through vertex x in Figure 4(b).

□

Lemma 3. *Let G be a triangulation whose vertex set V is a set of n points in the plane, and let γ be a real number with $0 < \gamma < \pi$. Let G' be the graph that is returned by algorithm* BDEGSUBGRAPH(G, γ). *Then the maximum degree of G' is at most $14 + \lceil \frac{2\pi}{\gamma} \rceil$.*

Proof. Let u be an arbitrary element of V. It follows from Lemma 1 that, before u is processed, it has at most 5 processed neighbors, say $u_1, u_2, \cdots, u_k$, where $k \leq 5$. In the worst case, each of its processed neighbors u_j can increase the degree of u by 3. (This is depicted in Figure 3, where the degree of the unprocessed vertex w is increased by 3 when v_i is processed.) This happens when, at the moment when u_j is processed, (i) u is the closest unprocessed neighbor of u_j in some cone with apex u_j, and (ii) both neighbors of u adjacent to u_j are also unprocessed. Hence, before u is processed, the degree of u (in G') is at most 15.

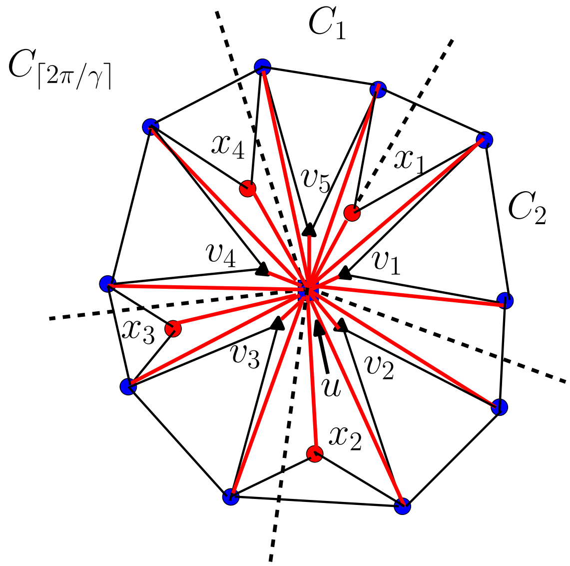


Figure 5: *$G(V, E)$ is the triangulation of given point set V. There exists a low-degree numbering where $v_1, \cdots, v_5$ occur after u. Therefore, when u is processed it has degree $14 + \lceil \frac{2\pi}{\gamma} \rceil$.*

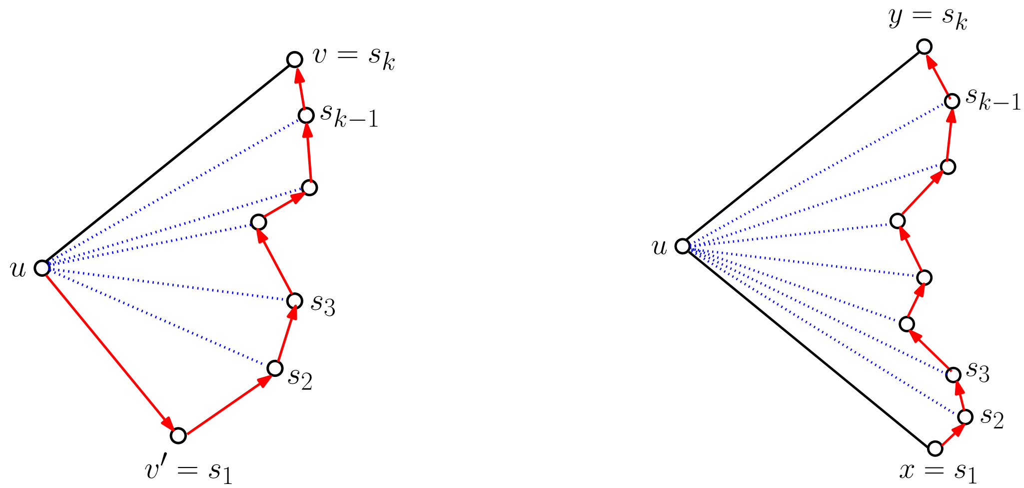
During the processing of u, the algorithm divides the plane into $\lceil \frac{2\pi}{\gamma} \rceil$ cones. For each cone, the algorithm adds one edge from u to its closest unprocessed neighbor. However, two of these cones (C_1 and C_2 in the algorithm) share an edge. Thus, during the processing of u, the degree of u (in G') is increased by at most $\lceil \frac{2\pi}{\gamma} \rceil - 1$.

After u has been processed, the degree of u does not change.

Thus, the maximum degree of any vertex of G' is at most $15 + \lceil \frac{2\pi}{\gamma} \rceil - 1 = 14 + \lceil \frac{2\pi}{\gamma} \rceil$. □

We now show that the upper bound on the maximum degree in Theorem 1 could be tight for a triangulation. Refer to Figure 5, let V be the point set in the plane. G is the triangulation of V and G' is the graph that is returned by algorithm BDegSubgraph(G, γ), where $0 < \gamma < \pi$. Then the maximum degree of G' is at most $14 + \lceil \frac{2\pi}{\gamma} \rceil$.

When processing vertex u, there could exist a low-degree numbering where $v_1, \cdots, v_5$ occur after u. By algorithm BDegSubgraph, v_i, $1 \le i \le 5$, is processed before u. Each v_i increases the degree of u by 3 before u is processed. During the processing of u, the algorithm divides the plane into $\lceil \frac{2\pi}{\gamma} \rceil$ cones. For each cone, the algorithm adds one edge from u to its closest unprocessed neighbor. C_1 and C_2 share an edge (u, x_1). Therefore, there exists a vertex u in G' having degree $15 + \lceil \frac{2\pi}{\gamma} \rceil - 1 = 14 + \lceil \frac{2\pi}{\gamma} \rceil$.



(a) An illustration of Lemma 4. (b) An illustration of Lemma 5.

Figure 6: *Illustrations of Lemma 4 and Lemma 5.*

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3 Bounded-degree spanners of the Delaunay triangulation

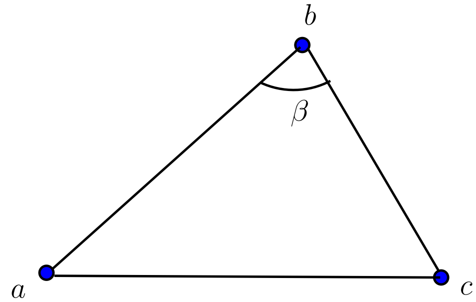
In this section, we assume that $G = (V, E)$ is the Delaunay triangulation of the point set V. Let $G' = (V, E')$ be the output of algorithm BDEGSUBGRAPH(G, γ). Lemma 3 gives an upper bound on the maximum degree of G'. In this section, we will prove that G' is a spanner of V, which will complete the proof of Theorem 1. Our analysis uses the following two lemmas, which state some basic properties of the Delaunay triangulation. These lemmas are illustrated in Figure 6.

Lemma 4. *[3] Let G be the Delaunay triangulation of V, let γ be a real number with $0 < \gamma < \pi/2$, and let u, v, and v' be three points of V, such that (u, v) and (u, v') are Delaunay edges and $\angle vuv' \leq \gamma$. Let $v' = s_1, s_2, \ldots, s_{k-1}, s_k = v$ be the Delaunay neighbors of u between v' and v, sorted in angular order around u. Assume that $|uv'| \leq |us_i|$ for all i with $1 \leq i \leq k$. Finally, let P_{uv} be the path $u, v', s_2, s_3, \ldots, s_{k-1}, v$ in G between u and v. Then, the length of P_{uv} is at most $t_\gamma |uv|$, where*

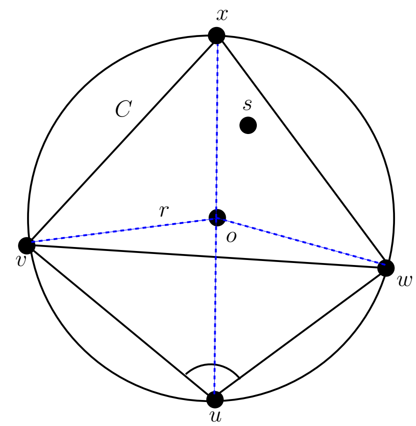
$$t_\gamma = \max\left\{\frac{\pi}{2}, 1 + \pi \sin\frac{\gamma}{2}\right\}.$$

Lemma 5. *[1] Let G be the Delaunay triangulation of V, and let u, x, and y be three points of V, such that (u, x) and (u, y) are Delaunay edges and $\angle xuy < \pi/2$. Let $x = s_1, s_2, \ldots, s_{k-1}, s_k = y$ be the Delaunay neighbors of u between x and y, sorted in angular order around u. Let Q_{xy} be the path $x, s_2, s_3, \ldots, s_{k-1}, y$ in G between x and y. Then, the length of Q_{xy} is at most*

$$\frac{\pi}{2}(|ux| + |uy|).$$



(a) An illustration of Lemma 6.



(b) An illustration of Lemma 7.

Figure 7: *Illustrations of Lemma 6 and Lemma 7.*

We also need the following two geometric lemmas.

Lemma 6. *Given a triangle Δ_{abc}, if $\angle abc \leq 2\pi/3$ and $|ab| \geq |bc|$, then $|ac| \leq \sqrt{3}|ab|$.*

Proof. Refer to Figure 7(a). In triangle Δ_{abc}, let $\beta = \angle abc$. By the Law of Cosines, we have

$$|ac| = \sqrt{|ab|^2 + |bc|^2 - 2|ab| \cdot |bc| \cos \beta}.$$

Since $|ab| \geq |bc|$ and $\beta \leq 2\pi/3$, we have

$$|ac| \leq \sqrt{2|ab|^2 - 2|ab| \cdot |bc| \cos(2\pi/3)} \leq \sqrt{2|ab|^2 + |ab||bc|} \leq \sqrt{3|ab|^2} = \sqrt{3}|ab|.$$

□

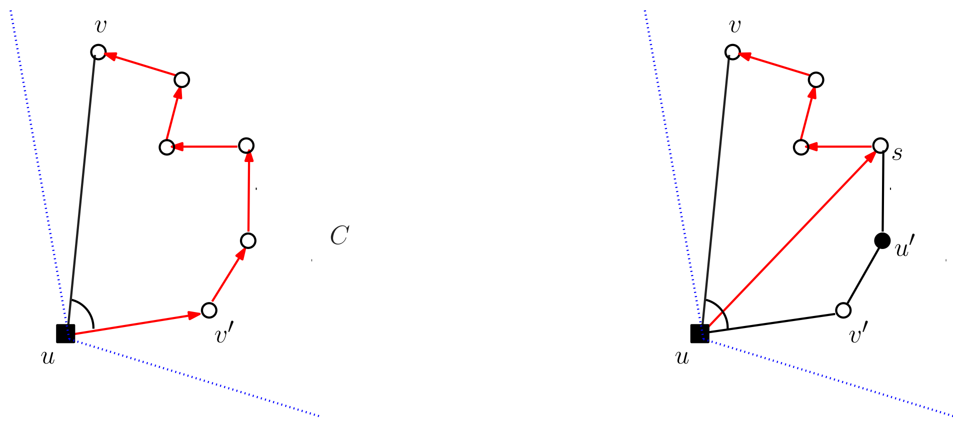
Lemma 7. *Let u, v, w, and s be four points in the plane, such that $\angle vuw \leq 2\pi/3$ and $|uv| \geq |uw|$, and let C be the circle through u, v, and w. Assume that s is contained in C. Then,*

$$|us| \leq 2|uv|.$$

Proof. Let o be the center of C and denote the radius of C by r; refer to Figure 7(b). Since s is inside C and u is on the boundary of C, we have $|us| \leq 2r$. Thus, we only need to show that $|uv| \geq r$.

Let x be the point on C for which $|ux| = 2r$. In the right triangles Δ_{uvx} and Δ_{uwx}, we have $|uv|^2 + |vx|^2 = |ux|^2$ and $|uw|^2 + |wx|^2 = |ux|^2$. Since $|uv| \geq |uw|$, we have $|vx| \leq |wx|$. It follows that $\angle vux \leq \angle xuw$. Since $\angle vuw = \angle vux + \angle xuw \leq 2\pi/3$, we have $\angle vux \leq \pi/3$. In the triangle Δ_{vuo}, we have $|ov| = |ou| = r$ and $\angle vuo = \angle vux \leq \pi/3$. It follows that $\angle vou \geq \pi/3$ and thus $|uv| \geq |ov| = |ou| = r$.

□



(a) All Delaunay-neighbors of u between v and v' are unprocessed.

(b) At least one Delaunay-neighbor of u between v and v' has been processed.

Figure 8: *Case* 1 *in the proof of Lemma 8.*

We can now prove the main result of this section, which completes the proof of Theorem 1.

Lemma 8. *Let $G = (V, E)$ be the Delaunay triangulation of V, let γ be a real number with $0 < \gamma \leq 2\pi/3$, and let $G' = (V, E')$ be the output of algorithm* BDegSubgraph(G, γ)*. Then, G' is a $\frac{4\pi\sqrt{3}}{9} \cdot t'$-spanner of V, where*

$$t' = \begin{cases} \max\left\{\frac{\pi}{2}, 1 + \pi \sin\frac{\gamma}{2}\right\} & \text{if } \gamma < \pi/2, \\ 1 + 2\sqrt{3} + 3\pi/2 + \pi \sin\frac{\pi}{12} & \text{if } \pi/2 \leq \gamma \leq 2\pi/3. \end{cases}$$

Proof. Recall that G is a $\frac{4\pi\sqrt{3}}{9}$-spanner of V; see Keil and Gutwin [7]. Also, G' is a subgraph of G. Therefore, it suffices to show that for each edge (u, v) of $G \setminus G'$, the graph G' contains a path between u and v, whose length is at most $t'|uv|$.

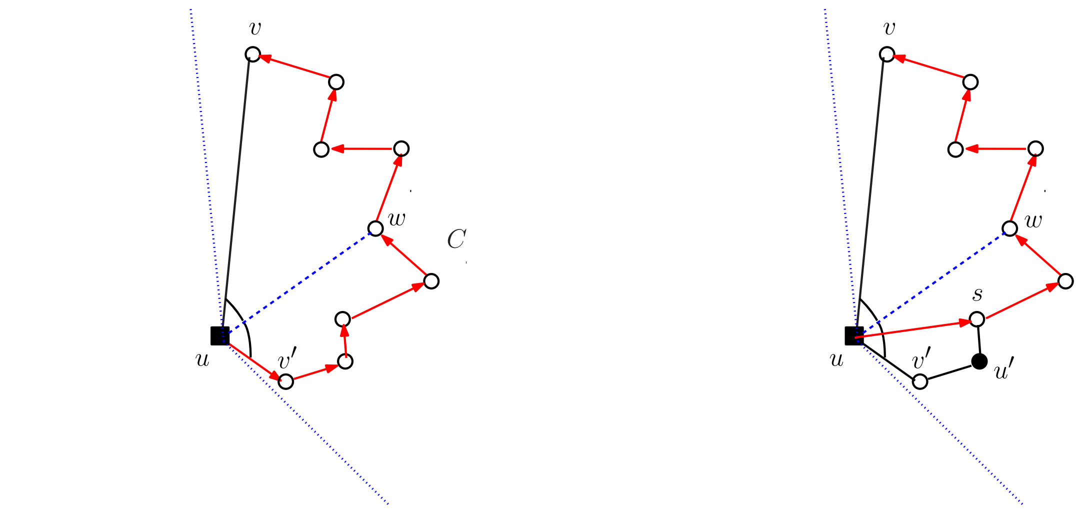
Throughout the rest of the proof, we fix an edge (u, v) of $G \setminus G'$. We may assume without loss of generality that u is processed before v. Let C be the cone with apex u and having angle at most γ that contains v and that is constructed when vertex u is processed. Let v' be the closest Delaunay-neighbor of u that is contained in C and that is unprocessed at the moment when u is processed. Thus, during the processing of u, the edge (u, v') is added to G'. We may assume without loss of generality that (u, v') is clockwise to the right of (u, v). Observe that $\angle vuv' \leq \gamma$.

We distinguish three cases.

Case 1**:** $\angle vuv' < \frac{\pi}{2}$.

Consider the path P_{uv} of Lemma 4 between u and v shown as a directed path through vertex v' in Figure 8(a). By Lemma 4, the length of this path is at most $t_\gamma|uv| \leq t'|uv|$.

We first assume that, at the moment when u is processed, all Delaunay-neighbors of u between v and v' are unprocessed. Then it follows from algorithm BDegSubgraph that P_{uv} is a path in G'.



(a) All Delaunay-neighbors of u between v and v' are unprocessed.

(b) At least one Delaunay-neighbor of u between v and v' has been processed.

Figure 9: *Case 2 in the proof of Lemma 8.*

Now assume that, at the moment when u is processed, at least one Delaunay-neighbor u' of u between v and v' has already been processed; refer to Figure 8(b). Let u' be the first such Delaunay-neighbor in clockwise order from (u, v). Observe that $u' \neq v$ and $u' \neq v'$. Let s be the Delaunay-neighbor of u such that s is between v and u' and $\triangle_{usu'}$ is a Delaunay-triangle. Then, by our choice of u', u' is processed before s. Thus, during the processing of u', the algorithm adds the edge (u, s) to G'. Let Q be the subpath of P_{uv} that starts at s and ends at v. Let Q' be the path obtained by concatenating the edge (u, s) and the subpath Q. It follows from algorithm BDegSubgraph that Q' is a path in G' between u and v. By the triangle inequality, the length of Q' is at most the length of P_{uv}. Thus, the length of Q' is at most $t_\gamma|uv| \leq t'|uv|$.

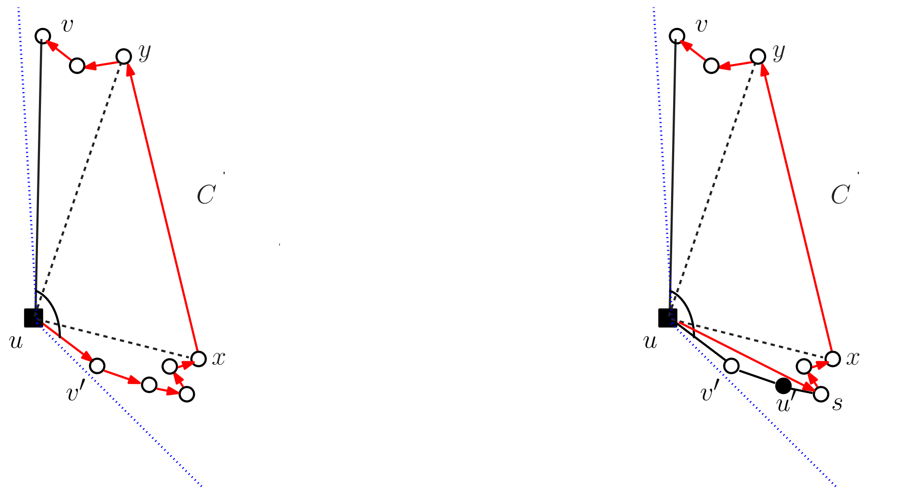
This concludes the analysis of Case 1. Observe that this case always holds if $\gamma < \pi/2$.

Case 2: $\angle vuv' \geq \frac{\pi}{2}$ and there is at least one Delaunay-neighbor w of u such that $\angle vuw < \frac{\pi}{2}$ and $\angle wuv' < \frac{\pi}{2}$.

Let P_{uw} be the path that starts at u, follows the edge (u, v'), and then follows the Delaunay-neighbors of u from v' to w. Let Q_{wv} be the path that starts at w, and follows the Delaunay-neighbors of u from w to v. Let P be the concatenation of P_{uw} and Q_{wv}. (Refer to Figure 9(a)).

The length of P is equal to the sum of the lengths of P_{uw} and Q_{wv}. Since $\angle wuv' < \frac{\pi}{2}$, it follows from Lemma 4 that the length of P_{uw} is at most $t_{\frac{\pi}{2}}|uw|$, where $t_{\frac{\pi}{2}} = \max\{\frac{\pi}{2}, 1 + \pi \sin \frac{\pi}{4}\} = 1 + \frac{\pi\sqrt{2}}{2}$. Since $\angle vuw < \frac{\pi}{2}$, it follows from Lemma 5 that the length of Q_{wv} is at most $\frac{\pi}{2}(|uv| + |uw|)$. Thus, the length of P is at most

$$\frac{\pi}{2}|uv| + \left(t_{\frac{\pi}{2}} + \frac{\pi}{2}\right)|uw|.$$



(a) All Delaunay-neighbors of u between v and v' are unprocessed. (b) At least one Delaunay-neighbor of u between v and v' has been processed.

Figure 10: *Case* 3 *in the proof of Lemma 8.*

We prove an upper bound on this quantity in terms of $|uv|$.

Recall that $(u, v), (u, w)$ and (u, v') are edges in the Delaunay triangulations of V. These three edges are also edges in the Delaunay triangulation of the point set $\{u, v, v', w\}$. It follows that $\Delta_{uvv'}$ is not a triangle in the latter triangulation and, therefore, w is in the circle through u, v and v'. It then follows from Lemma 7 that $|uw| \leq 2|uv|$. Thus, the length of P is at most

$$\frac{\pi}{2}|uv| + \left(t_{\frac{\pi}{2}} + \frac{\pi}{2}\right) \cdot (2|uv|) = (2\ t_{\frac{\pi}{2}} + 3\pi/2)|uv| \leq t'|uv|.$$

Assume that, at the moment when u is processed, all Delaunay-neighbors of u between v and v' are unprocessed. Then it follows from algorithm BDegSubgraph that P is a path in G' between u and v. If at least one Delaunay-neighbor u' of u between v and v' has already been processed (refer to Figure 9(b)), then, by a similar argument as in Case 1, G' contains a path between u and v which is obtained by shortcutting P.

Case 3: $\angle vuv' \geq \frac{\pi}{2}$ and there is no Delaunay-neighbor w of u such that $\angle vuw < \frac{\pi}{2}$ and $\angle wuv' < \frac{\pi}{2}$.

In this case, there exist two Delaunay-edges (u, x) and (u, y) such that $\angle yuv' \geq \frac{\pi}{2}$, $\angle vux \geq \frac{\pi}{2}$, and (x, y) is a Delaunay-edge. (Refer to Figure 10(a).)

Let P_{ux} be the path that starts at u, follows the edge (u, v'), and then follows the Delaunay-neighbors of u from v' to x. Let Q_{yv} be the path that starts at y, and follows the Delaunay-neighbors of u from y to v. Let P be the concatenation of P_{ux}, the edge (x, y), and Q_{yv}. Then the length of P is equal to the sum of the lengths of P_{ux}, Q_{yv}, and $|xy|$. Since $\angle vux \geq \frac{\pi}{2}$ and $\angle vuv' \leq \frac{2\pi}{3}$, we have $\angle xuv' \leq \frac{\pi}{6}$. Therefore, it follows from Lemma 4 that the length of P_{ux} is at most $t_{\frac{\pi}{6}}|ux|$, where $t_{\frac{\pi}{6}} = \max\{\frac{\pi}{2}, 1 + \pi \sin \frac{\pi}{12}\} = 1 + \pi \sin \frac{\pi}{12}$. Since $\angle vux < \frac{\pi}{2}$, we have $\angle vuy < \frac{\pi}{2}$. Thus, by Lemma 5, the length of Q_{yv} is at most

$\frac{\pi}{2}(|uv| + |uy|)$. Thus, the length of P is at most

$$t_{\frac{\pi}{6}}|ux| + |xy| + \frac{\pi}{2}(|uv| + |uy|).$$

We will prove an upper bound on this quantity in terms of $|uv|$.

Since Δ_{uvx} is not a triangle in the Delaunay triangulation of $\{u, v, v', x\}$, x is in the circle through u, v and v'. It then follows from Lemma 7 that $|ux| \leq 2|uv|$. By a similar argument, y is in the circle through u, v and v' and, again by Lemma 7, $|uy| \leq 2|uv|$. Let $z = \max(|ux|, |uy|)$. By Lemma 6, we have $|xy| \leq \sqrt{3}z \leq 2\sqrt{3}|uv|$. Thus, the length of P is at most

$$t_{\frac{\pi}{6}}(2|uv|) + 2\sqrt{3}|uv| + \frac{\pi}{2}(3|uv|) = (1 + 2\sqrt{3} + 3\pi/2 + \pi \sin\frac{\pi}{12})|uv| \leq t'|uv|.$$

If, at the moment when u is processed, all Delaunay-neighbors of u between v and v' are unprocessed (see Figure 10(a)), then P is a path in G' between u and v. Otherwise, at least one Delaunay-neighbor u' of u between v and v' had already been processed (see Figure 10(b)). In this case, by a similar argument as in Case 1, G' contains a path between u and v which is obtained by shortcutting P.

□

We now show that the upper bound on the maximum degree in Theorem 1 is almost tight for Delaunay triangulations. Refer to Figure 11, let V be the point set in the plane. G is the Delaunay triangulation of V and G' is the graph that is returned by algorithm BDegSubgraph(G, γ), where $\gamma = 2\pi/3$. Then the maximum degree of G' is at most $14 + \lceil\frac{2\pi}{2\pi/3}\rceil = 14 + 3 = 17$.

When processing vertex u, there could exist a low-degree numbering where $v_1, \cdots, v_5$ occur after u. By algorithm BDegSubgraph, v_i, $1 \leq i \leq 5$, is processed before u. Each v_i increases the degree of u by 3 before u is processed. Since x is the closest unprocessed neighbor of u, the algorithm adds one edge (u, x) when processing u. Therefore, there exists a vertex u in G' having degree 16. Therefore, it seems that there exists very little chance to further improve the degree bound by applying similar techniques of this paper.

4 Bounded-degree spanners of a diamond triangulation

Let V be a set of n points in the plane, let α be a real number with $0 < \alpha < \frac{\pi}{2}$, and let $G = (V, E)$ be a triangulation that satisfies the α-diamond property; see Section 1 for the definition of this property. In this section, we complete the proof of Theorem 2, by showing that the output $G' = (V, E')$ of algorithm BDegSubgraph(G, α) is a t-spanner, for some value t that only depends on α. The key to the proof is the following lemma which generalizes Lemma 4.

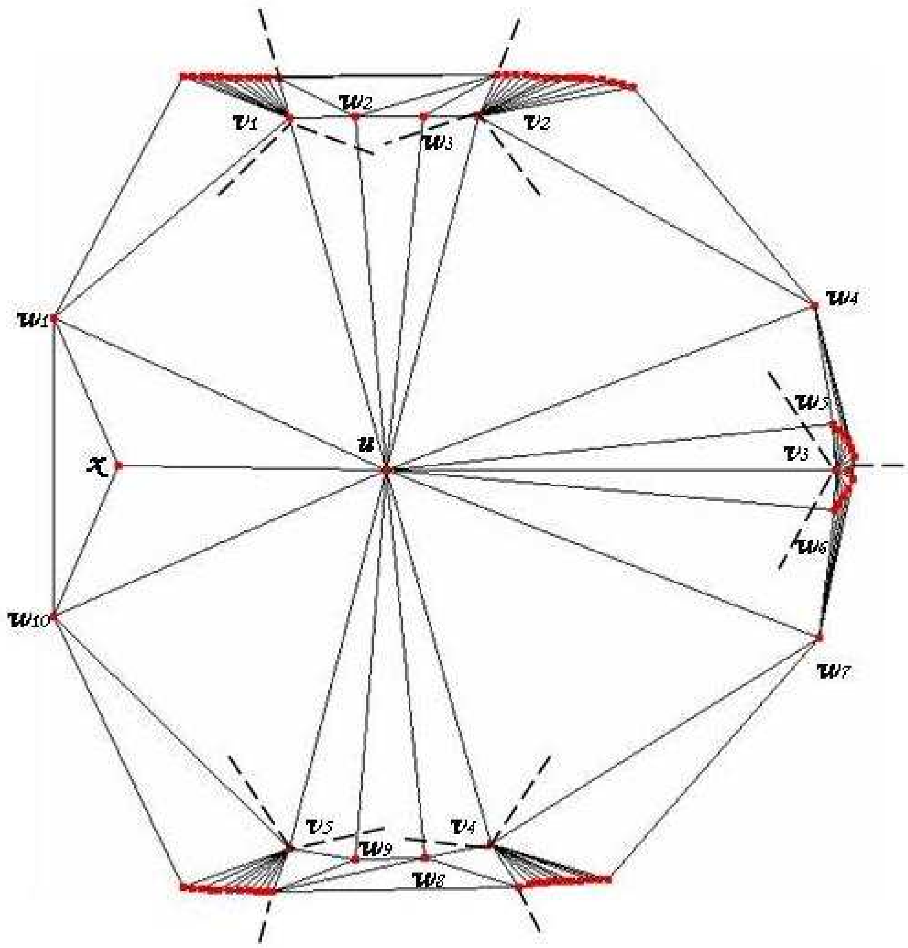


Figure 11: $G(V, E)$ *is the Delaunay triangulation of given point set* V*. There exists a low-degree numbering where* $v_1, \cdots, v_5$ *occur after* u*. Therefore, when* u *is processed it has degree* 16.

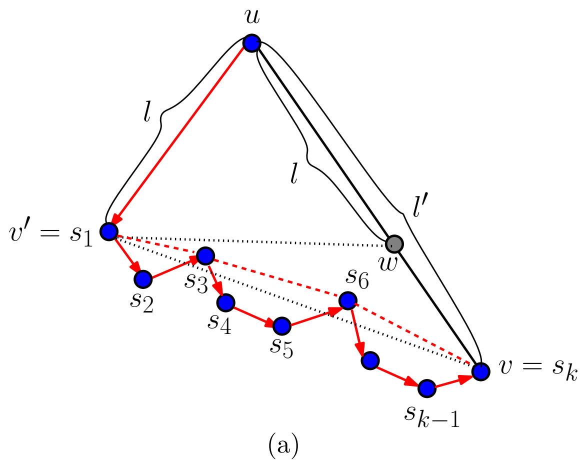


Figure 12: *An illustration of Lemma 9.*

Lemma 9. *Let $G = (V, E)$ be a triangulation of the point set V, and let α be a real number with $0 < \alpha < \frac{\pi}{2}$, such that G satisfies the α-diamond property. Let u, v and v' be three points of V, such that (u, v) and (u, v') are edges of G and $\angle vuv' \leq \alpha$. Let $v' = s_1, s_2, \cdots, s_k = v$ be the neighbors of u in G between v' and v, sorted in angular order around u. Assume that $|uv'| \leq |us_i|$ for all i with $1 \leq i \leq k$. Let P_{uv} be the path $u, v', s_2, s_3, \cdots, s_{k-1}, v$. Then, P_{uv} is a path in G between u and v, whose length is at most $t'_\alpha|uv|$, where*

$$t'_\alpha = 1 + \frac{2(\pi - \alpha)}{\alpha \sin \frac{\alpha}{4}} \cdot \max\left\{1, 2\sin\frac{\alpha}{2}\right\}.$$

Proof. Let Q be the path $v' = s_1, s_2, s_3, \cdots, s_{k-1}, s_k = v$. Let R be the shortest path between v' and v that is completely contained in the polygon with vertices $u, v', s_2, \ldots, s_{k-1}, v$. (In Figure 12(a), $R = (v', s_3, s_6, v)$.) Consider any edge (s_i, s_j) of R, and let $Q' = (s_i, s_{i+1}, \ldots, s_j)$ be the corresponding subpath of Q. Observe that Q' is completely on one side of the line segment $s_i s_j$. (An example is the subpath (s_3, s_4, s_5, s_6) in Figure 12(a).)

Consider an arbitrary vertex s_m of Q' with $i < m < j$. Since (u, s_m) is an edge in G, and since G satisfies the α-diamond property, at least one of the two isosceles triangles Δ_1 and Δ_2 with base (u, s_m) and base angle α does not contain any point of V in its interior. Let a_1 and a_2 be the two apices of these triangles; see Figure 12(b). We may assume without loss of generality that the triangle with apex a_1 is empty. We claim that a_1 and u are on the same side of the line through s_i and s_j. Indeed, assume that a_1 is "below" (s_i, s_j), then, since $\angle a_1 u s_m = \alpha$, the triangle $\triangle_{ua_1s_m}$ must contain the vertex s_i, which is a contradiction.

Thus, for each vertex s_m on Q', with $i < m < j$, the apex of the empty α-triangle of the edge (u, s_m) is on the same side of the line through $s_i s_j$ as u. Using results of Das and Joseph [5] and Lee [8], it then follows that the length of the subpath Q' is at most $c_\alpha|s_i s_j|$, where $c_\alpha = \frac{2(\pi-\alpha)}{\alpha\cdot\sin(\alpha/4)}$.

Hence, the length of Q is at most c_α times the length of R. We now prove an upper bound on the length of R.

Let w be the point on the edge (u,v) such that $|uv'| = |uw|$. We define $\ell = |uv'|$ and $\ell' = |uv|$, and observe that $\ell' \geq \ell$. Since the length of each edge (u, s_m), $1 \leq m \leq k$, is at least ℓ, no point could be inside the triangle $\Delta_{uv'w}$. Since R is the shortest path between v' and v that is completely contained in the polygon with vertices $u, v', s_2, \ldots, s_{k-1}, v$, by convexity, each point in R must be inside the triangle $\Delta_{uv'v}$. Thus, the path R is contained in the triangle $\Delta_{v'vw}$. Therefore, by convexity, the length of R is at most $|vw|+|wv'|$. Denote $\angle vuv'$ by θ and observe that $\theta \leq \alpha$. Then, we have

$$\begin{aligned} |vw| + |wv'| &= \ell' - \ell + 2\ell \sin\frac{\theta}{2} \\ &\leq \ell' - \ell + 2\ell \sin\frac{\alpha}{2} \\ &= \ell' - \left(1 - 2\sin\frac{\alpha}{2}\right)\ell. \end{aligned}$$

Thus, if $1 - 2\sin\frac{\alpha}{2} \geq 0$, then $|vw| + |wv'| \leq \ell' = |uv|$. On the other hand, if $1 - 2\sin\frac{\alpha}{2} < 0$, then

$$|vw| + |wv'| \leq \ell' + \left(2\sin\frac{\alpha}{2} - 1\right)\ell \leq \ell' + \left(2\sin\frac{\alpha}{2} - 1\right)\ell' = 2\sin\frac{\alpha}{2}\,|uv|.$$

This shows that,

$$|vw| + |wv'| \leq \max\left\{1, 2\sin\frac{\alpha}{2}\right\} \cdot |uv|.$$

We conclude that the length of the path P_{uv} in the lemma is at most $|uv'|$ plus c_α times the length of R, which in turn is at most $|uv|$ plus $c_\alpha \cdot \max\{1, 2\sin\frac{\alpha}{2}\}|uv|$. □

We are now able to prove that algorithm BDegSubgraph(G, α) computes a spanner of the diamond triangulation G and thereby completing the proof of Theorem 2.

Lemma 10. *Let $G = (V, E)$ be a triangulation and let α be a real number with $0 < \alpha < \frac{\pi}{2}$, such that G satisfies the α-diamond property. Let $G' = (V, E')$ be the output of algorithm* BDegSubgraph(G, α). *Then, then G' is a t-spanner of V, where*

$$t = t'_\alpha \cdot \frac{8(\pi - \alpha)^2}{\alpha^2 \sin^2\frac{\alpha}{4}},$$

and t'_α is given in Lemma 9.

Proof. Let (u,v) be an arbitrary edge of $G \setminus G'$. Using the same proof technique as in Theorem 1, and using Lemma 9, it can be shown that G' contains a path between u and v whose length is at most $t'_\alpha|uv|$. (Since $0 < \alpha < \frac{\pi}{2}$, only Case 1 in the proof of Theorem 1 needs to be considered.) Since Das and Joseph [5] and Lee [8] have shown that G is a t'-spanner of V, for $t' = \frac{8(\pi-\alpha)^2}{\alpha^2 \sin^2\frac{\alpha}{4}}$, the lemma follows. □

5 Bounded-degree spanners of the unit-disk graph

Let V be a set of n points in the plane, and let $UDel$ be the graph with vertex set V, and whose edge set is the set of all edges in the Delaunay triangulation of V whose length is at most one. In general, $UDel$ is not a triangulation, even though it is a plane graph. For simplicity, we assume that $UDel$ is connected. If $UDel$ is not connected, then we consider each connected component of $UDel$ separately. In this section, we show that we can modify algorithm BDEGSUBGRAPH to construct a bounded-degree spanner of the unit-disk graph. The modified algorithm is given in Figure 13. In this algorithm, $N_G(v)$ denotes the set of *neighbors* of the vertex v in a graph G, i.e. $N_G(v) = \{w \in V : (v, w) \text{ is an edge in } G\}$.

Observe that in lines 7 and 25 of this algorithm, edges may be added to the graph G' which are not in $UDel$. Thus, the algorithm computes a subgraph G' of the unit-disk graph which is not necessarily a subgraph of $UDel$. As illustrated in Figure 14, edges (s_1, s_2) and (s_5, s_6) which are added in line 7 and edge (s_3, s_4) which is added in line 25 are not edges of $UDel$.

We assume that the Delaunay triangulation is stored in a doubly-connected edge list. Then, for any vertex v, the time spend for adding edges in line 7 is proportional to the degree of v in the Delaunay triangulation of V. We can also obtain the vertices in $N_{G^*}(v)$ in G^*, sorted in angular order around v, in time proportional to the degree of v in G^*. It will be shown in Lemma 11 that G^* is a plane graph and thus contains $O(n)$ edges. These observations, together with Lemma 1, imply that the running time of algorithm BDEGUDEL is $O(n)$.

We show that, when $0 < \gamma \leq \pi/3$, the graph G' is a plane bounded-degree spanner of the unit-disk graph of V.

Lemma 11. *Let $G = (V, E)$ be the Delaunay triangulation of V, and let γ be a real number with $0 < \gamma \leq \pi/3$. Let $UDG(V)$ be the unit-disk graph on V. Let G' be the graph that is returned by algorithm* BDEGUDEL(G, γ). *Then, G' is a plane subgraph of $UDG(V)$ and the maximum degree of G' is at most $14 + \lceil \frac{2\pi}{\gamma} \rceil$.*

Proof. Since G is the Delaunay triangulation of V, G is planar. Therefore, it suffices to show that for each edge (u, v) of G' but not in G, the planarity of G' holds and $|uv| \leq 1$. Note that edge (u, v) can only be added to G' in lines 7 and 25 of algorithm BDEGUDEL(G, γ). In line 7, edge $(u_s, u_{(s+1) \bmod d})$ is only intersected by those Delaunay edges between edges (v, u_s) and $(v, u_{(s+1) \bmod d})$ whose lengths are greater than one. By algorithm BDEGUDEL(G, γ), these Delaunay edges are not added to E' and thus planarity holds. In line 25, w_k and $w_{(k+1) \bmod d}$ are consecutive unprocessed neighbors of v in G^*. If (v, w) is an edge which is intersected by edge $(w_k, w_{(k+1) \bmod d})$, (v, w) could not be one of those edges added in line 7 for $|vw| > 1$. Therefore, edge $(w_k, w_{(k+1) \bmod d})$ is only intersected by those Delaunay edges between edges (v, w_k) and $(v, w_{(k+1) \bmod d})$ whose lengths are greater than one. By algorithm BDEGUDEL(G, γ), these Delaunay edges are not added to E' and thus planarity holds too. Note that the length of any newly added edge is at most one. Therefore, G' is a plane subgraph of $UDG(V)$.

```
Algorithm BDegUDel(G, γ)
Input: The Delaunay triangulation G = (V, E) whose vertex set V is a set of n points in
       the plane, and a real number γ with 0 < γ < π/3.
1.  let UDel be the subgraph of G consisting of all edges of length at most one;
2.  let G* = UDel;
3.  for each v ∈ V
4.      do let u_0, u_1, ···, u_{d−1} be the vertices in N_{UDel}(v), ordered in clockwise order around
           v;
5.         for s = 0 to d − 1
6.                 if (v, u_s, u_{(s+1) mod d}) is not a triangle in G and |u_s u_{(s+1) mod d}| ≤ 1
7.                    then add the edge (u_s, u_{(s+1) mod d}) to G*;
8.  compute a low-degree numbering (v_1, v_2, ···, v_n) of G*;
9.  label each vertex of V as "unprocessed";
10. E' = ∅;
11. for i = n downto 1
12.     do if v_i has "unprocessed" Delaunay neighbors in UDel
13.           then compute the closest "unprocessed" Delaunay neighbor x of v_i in UDel;
14.                divide the plane into cones C_1, ..., C_{⌈2π/γ⌉} with apex v_i and angle at
                   most γ such that the segment v_i x is on the boundary between C_1 and
                   C_2;
15.                add the edge (v_i, x) to E'
16.           else go to line 26
17.        for each cone C ∉ {C_1, C_2}
18.            do compute the closest "unprocessed" Delaunay neighbor w in C ∩
                  N_{UDel}(v_i);
19.               if w exists
20.                  then add the edge (v_i, w) to E'
21.        let w_0, w_1, ···, w_{d−1} be the vertices in N_{G*}(v_i), ordered in clockwise order around
           v_i;
22.        for k = 0 to d − 1
23.                if w_k and w_{(k+1) mod d} are both "unprocessed"
24.                       and |w_k w_{(k+1) mod d}| ≤ 1
25.                   then add the edge (w_k, w_{(k+1) mod d}) to E';
26.        label v_i as "processed";
27. return the graph G' = (V, E')
```

Figure 13: *The algorithm that computes a bounded-degree spanner of the unit-disk graph.*

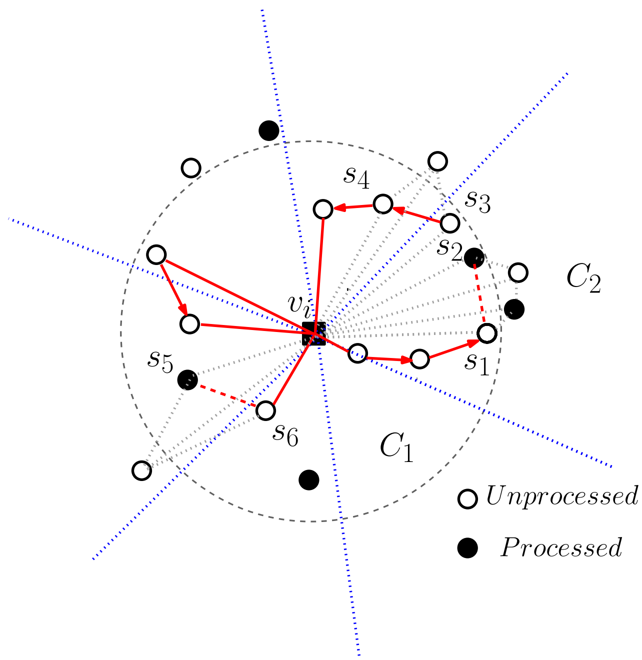


Figure 14: *An illustration of algorithm* BDEGUDEL(G,γ)*, for* $\gamma = \pi/3$*, when processing vertex* v_i*. The figure shows* v_i*, all vertices in* $N_{G^*}(v_i)$ *and all Delaunay neighbors of* v_i *whose distance from* v_i *is greater than one. When processing* v_i*, the algorithm adds the dashed, solid and arrowed edges to the graph* G'*.*

Let u be an arbitrary element of V. Similar to Lemma 1, before u is processed, it has at most 5 processed neighbors, say $u_1, u_2, \cdots, u_k$, where $k \leq 5$. In the worst case, each of its processed neighbors u_j can increase the degree of u by 3. Hence, before u is processed, the degree of u (in G') is at most 15.

During the processing of u, the algorithm divides the plane into $\lceil \frac{2\pi}{\gamma} \rceil$ cones. For each cone, the algorithm adds one edge from u to its closest unprocessed neighbor. However, two of these cones (C_1 and C_2 in the algorithm) share an edge. Thus, during the processing of u, the degree of u (in G') is increased by at most $\lceil \frac{2\pi}{\gamma} \rceil - 1$.

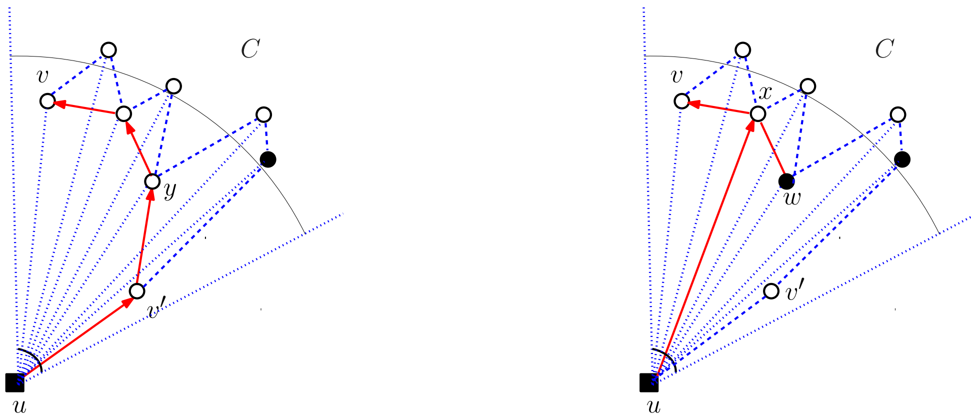
After u has been processed, the degree of u does not change.

Thus, the maximum degree of any vertex of G' is at most $15 + \lceil \frac{2\pi}{\gamma} \rceil - 1 = 14 + \lceil \frac{2\pi}{\gamma} \rceil$. □

We now prove that the output of algorithm BDEGUDEL(G,γ) is a spanner of the unit-disk graph. This will complete the proof of Theorem 3.

Lemma 12. *Let* $G = (V,E)$ *be the Delaunay triangulation of* V*, and let* γ *be a real number with* $0 < \gamma \leq \pi/3$*. Let* $UDG(V)$ *be the unit-disk graph on* V*. Let* G' *be the graph that is returned by algorithm* BDEGUDEL(G,γ)*. Then,* G' *is a* $\frac{4\pi\sqrt{3}}{9} \cdot t_\gamma$*-spanner of* $UDG(V)$*, where*

$$t_\gamma = \max\left\{\frac{\pi}{2}, 1 + \pi \sin\frac{\gamma}{2}\right\}$$



(a) All the neighbors of u between v and v' are unprocessed.

(b) At least one neighbor of u between v and v' has been processed.

Figure 15: *Illustrations of the proof of Lemma 12.*

Proof. The proof is very similar to the one of Lemma 8. Recall that $UDel$ is a $\frac{4\pi\sqrt{3}}{9}$-spanner of $UDG(V)$; see Bose *et al.* [2]. Therefore, it suffices to show that for each edge (u, v) of G' but not in $UDel$, the graph G' contains a path between u and v, whose length is at most $t_\gamma|uv|$. Throughout the rest of the proof, we fix an edge (u, v) of G' that is not in $UDel$. We may assume without loss of generality that u is processed before v. Let C be the cone with apex u and having angle at most γ that contains v and that is constructed when vertex u is processed. Let v' be the closest Delaunay neighbor of u in G^* of algorithm BDegUDel(G, γ) that is contained in C and that is unprocessed at the moment when u is processed. Thus, during the processing of u, the edge (u, v') is added to G'. We may assume without loss of generality that (u, v') is clockwise to the right of (u, v). Observe that $\angle vuv' \leq \gamma$.

Consider the path P_{uv} of Lemma 4 between u and v shown as a dashed path through vertex v' in Figure 15(a). By Lemma 4, the length of this path is at most $t_\gamma|uv|$, where $t_\gamma = \max\{\frac{\pi}{2}, 1 + \pi \sin \frac{\gamma}{2}\}$.

We first assume that, at the moment when u is processed, all neighbors of u between v and v' are unprocessed. Then it follows from algorithm BDegUDel that there is a path Q_{uv} between u and v shown as a solid directed path through vertex v' in Figure 15(a). If all the vertices in Q_{uv} are Delaunay neighbors of u in G, Q_{uv} will shortcut P_{uv} and the length of this path is at most $t_\gamma|uv|$. Assume that there is a vertex $y \in Q_{uv}$, which is not a Delaunay neighbor of u. By algorithm BDegUDel, there is a Delaunay edge (s, s') that intersects the edge (u, y), where s and s' are two consecutive Delaunay neighbors of u in G, $|us| \leq 1$, $|us'| \leq 1$ and $|ss'| > 1$. Thus, the angle $\angle sus'$ is larger than $\pi/3$ and C contains at most one of s and s'. Without loss of generality, assume that $s' \notin C$; refer to Figure 16. Notice that (u, v') is clockwise to the right of (u, s'). Therefore, we have $v' \notin C$, which contradicts the fact that $v' \in C$. Therefore, all the vertices in Q_{uv} are Delaunay neighbors of u in $UDel$.

Now assume that, at the moment when u is processed, at least one neighbor w of u

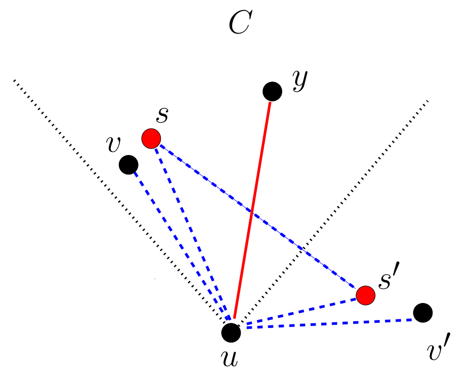


Figure 16: *An illustration of the proof of Lemma 12.*

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between v and v' has already been processed; refer to Figure 15(b). Let w be the first such neighbor in clockwise order from (u, v). Observe that $w \neq v$ and $w \neq v'$. Let x be the neighbor of u such that x is between v and w and $\triangle_{uxw} \in G^*$. Then, by our choice of w, w is processed before x. Thus, during the processing of w, the algorithm adds the edge (u, x) to G'. Let Q be the subpath of Q_{uv} that starts at x and ends at v. Let Q' be the path obtained by concatenating the edge (u, x) and the subpath Q. It follows from algorithm BDegUDel that Q' is a path in G' between u and v. By the triangle inequality, the length of Q' is at most the length of Q_{uv}. Thus, the length of Q' is at most $t_\gamma|uv|$.

□

6 Concluding remarks

We have shown that the Delaunay triangulation and, in fact, any triangulation satisfying the diamond property, contains a t-spanner of bounded degree, for some constant t. The smallest degree bound that we obtained is 17 (see Theorem 1). We leave open the problem of further reducing the degree. To be more precise, we pose the following open problems:

1. What is the smallest integer d, such that there exists a constant t (depending on d), such that the Delaunay triangulation of any point set V, or any triangulation satisfying the diamond property, contains a subgraph which is a t-spanner of V?

2. What is the smallest integer d, such that there exists a constant t (depending on d), such that for every point set V, a plane t-spanner of V exists?

We pose the same open problems for spanners of the unit-disk graph.

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